Spaces of Symmetric Matrices of Bounded Rank

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ABSTRACT

Let $\mathcal{S}_n(F)$ denote the space of all $n \times n$ symmetric matrices over a field F. Let t be a positive integer such that $t < n$. A subspace W of $\mathcal{S}_n(F)$ is said to be a $\bar{t}$-subspace provided that the rank of every matrix in W is bounded by t. Meshulam showed, under the assumption $|F| \geqslant n + 1$, that the maximal dimension of a $\bar{t}$-subspace of $\mathcal{S}_n(F)$ is given by

$$\max\left\{\binom{t+1}{2}, \binom{k+1}{2} + k(n-k)\right\} \qquad \text{if} \quad t = 2k,$$
$$\max\left\{\binom{t+1}{2}, \binom{k+1}{2} + k(n-k) + 1\right\} \quad \text{if} \quad t = 2k+1.$$

Provided that we also assume char $F \neq 2$, we show here that any $\bar{t}$-subspace of $\mathcal{S}_n(F)$ of maximal dimension is congruent to

$$W_1(n,t) = \left\{A \in \mathcal{S}_n(F) : a_{ij} = 0 \text{ if } i > t \text{ or } j > t\right\},$$

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or

$$W_2(n,t) = \begin{cases} \{A \in \mathscr{S}_n(F) : a_{ij} = 0 \text{ if } i > k \text{ and } j > k\}, & \text{if} \quad t = 2k, \\ \{A \in \mathscr{S}_n(F) : a_{ij} = 0 \text{ if } i > k \text{ and } j > k, \\ \quad \text{and } (i,j) \neq (k+1, k+1)\}, & \text{if} \quad t = 2k+1. \end{cases}$$

Which of the two possibilities occurs depends on the values of n and t.

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1. INTRODUCTION

Let F be a field and let m, n be positive integers. Denote by $F^{m,n}$ the space of all $m \times n$ matrices over F, and by $\mathscr{S}_n(F)$ the space of all $n \times n$ symmetric matrices over F. Let t be a positive integer such that $t < \min\{m, n\}$. A subspace W of $F^{m,n}$ is said to be a $\bar{t}$-subspace provided that the rank of every matrix in W is bounded above by t. $\bar{t}$-subspaces of $\mathscr{S}_n(F)$ are defined analogously (here $t < n$).

The problem of determining the structure of $\bar{t}$-subspaces is very interesting and difficult. Several works on the subject have appeared, but much more has to be studied. Here we consider the easier problem of giving an upper bound for the dimension of a $\bar{t}$-subspace. Flanders [4] showed that if W is a $\bar{t}$-subspace of $F^{m,n}$, then

$$\dim W \leqslant \max\{mt, nt\}. \tag{1.1}$$

Flanders assumed that $|F| \geqslant t + 1$. He also characterized the $\bar{t}$-subspaces of $F^{m,n}$ for which equality holds in (1.1), provided also that $\operatorname{char} F \neq 2$. Atkinson and Lloyd [1] extended Flanders's work, and managed to remove the assumption on $\operatorname{char} F$ in the case of equality in (1.1).

Meshulam [5] reproduced Flanders's result (assuming $m = n$), removing all restrictions on F. His approach is combinatorial. Later, Meshulam [6] computed the maximal dimension of a $\bar{t}$-subspace of $\mathscr{S}_n(F)$, assuming $|F| \geqslant n + 1$ (see Theorem 5.1). Meshulam's interesting approach in the symmetric case is of the same flavor as in the nonsymmetric case, and some of it will be reflected in our work.

Meshulam did not characterize the $\bar{t}$-subspaces of $\mathscr{S}_n(F)$ for which the maximal dimension is attained. It is our purpose here to obtain this characterization.

The paper is organized as follows: Section 2 contains some definitions and notation. Section 3 contains graph theoretic preliminaries. Section 4 contains some matrix theoretic lemmas which are used in the proof of the main result. Sections 5 and 6 constitute the main part of our work. In Section 5 we show that a $\bar{t}$-subspace of $\mathscr{S}_n(F)$ of maximal dimension must possess a basis of a special type. This is done using Meshulam's approach. In Section 6 we prove our main result, using linear algebraic methods.

2. SOME DEFINITIONS AND NOTATION

Let $A \in F^{m,n}$. We denote by A^t the transpose of A. If A is square, we denote its determinant by det A. $A[\alpha|\beta]$ denotes the submatrix of A based on row indices from α and column indices from β. $A[\alpha|\alpha]$ is denoted by $A[\alpha]$. The rank of A is denoted by $\rho(A)$. We let E_{ij} denote the matrix all of whose entries are 0 except the entry in the i, j position, which is 1.

Given any subspace W of $\mathscr{S}_n(F)$ and an invertible $S \in F^{n,n}$, define $SWS^t = \{SAS^t : A \in W\}$. Given two subspaces W and W' of $\mathscr{S}_n(F)$, we say W' is *congruent* to W if there exists an invertible $S \in F^{n,n}$ such that $W' = SWS^t$.

Let t be a positive integer such that $t < n$. We define

$$\alpha(n,t) = \begin{cases} \max\left\{\binom{t+1}{2}, \binom{k+1}{2} + k(n-k)\right\} & \text{if } t = 2k, \\ \max\left\{\binom{t+1}{2}, \binom{k+1}{2} + k(n-k) + 1\right\} & \text{if } t = 2k+1. \end{cases}$$

REMARK 2.1. It is easy to check that

(a) if $t = 2k$, then

$$\alpha(n,t) = \binom{t+1}{2} \qquad \text{if} \quad n < 2.5k + 0.5,$$

$$\alpha(n,t) = \binom{k+1}{2} + k(n-k) \qquad \text{if} \quad n > 2.5k + 0.5;$$

$$\alpha(n,t) = \binom{t+1}{2} = \binom{k+1}{2} + k(n-k) \qquad \text{if} \quad n = 2.5k + 0.5;$$

(b) if $t = 2k + 1$, then

$$\alpha(n,t) = \binom{t+1}{2} \qquad \text{if} \quad n < 2.5k + 2.5,$$

$$\alpha(n,t) = \binom{k+1}{2} + k(n-k) + 1 \qquad \text{if} \quad n > 2.5k + 2.5,$$

$$\alpha(n,t) = \binom{t+1}{2} = \binom{k+1}{2} + k(n-k) + 1 \qquad \text{if} \quad n = 2.5k + 2.5.$$

The following lemma is readily verified.

LEMMA 2.1. *Let $r \geqslant 2$ and let $x_1, x_2, \dots, x_r$ be natural numbers. Then*

$$\sum_{i=1}^{r} \binom{x_i}{2} \leqslant \binom{\sum_{i=1}^{r} x_i - r + 1}{2},$$

and equality holds if and only if at least $r - 1$ of the given numbers are equal to 1.

Finally, let $N = \{1, 2, \dots, n\}$.

3. GRAPH THEORETIC PRELIMINARIES

All graphs considered here are undirected, but may have loops. Thus, an edge e of a graph whose vertex set is N is a nonempty subset of N of cardinality $\leqslant 2$, and we write

$$|e| = \begin{cases} 1 & \text{if} \quad e = \{u\} \text{ for some } u \in N, \\ 2 & \text{if} \quad e = \{u, v\} \text{ for some } u, v \in N,\ u \neq v \end{cases}$$

(so $|e| = 1$ if and only if e is a loop).

A graph G is said to be *simple* if it has no loops.

We assume that the reader is familiar with some of the basic notions of graph theory. Suppose that G is a simple graph with vertex set N. If $S \subset N$, we let $G[S]$ denote the subgraph of G induced by S, and $G \setminus S$ denote the subgraph of G induced by $N \setminus S$. Let $C_1(G)$ denote the number of odd components of G.

Let B be a graph (possibly with loops) with vertex set N. The number of edges of B is denoted by $|B|$. A set B' of mutually disjoint edges of B is called a *matching* of B. Define

$$\mu(B) = \max\left\{ \sum_{e \in B'} |e| : B' \text{ is a matching of } B \right\}. \tag{3.1}$$

If G is a simple graph on N, let $\nu(G)$ be the maximum number of edges in any matching. Then clearly,

$$\mu(G) = 2\nu(G).$$

Given any positive integer t such that $t \leqslant n$, let

$$u(n,t) = \max\{|B| : B \text{ is a graph on } N \text{ with } \mu(B) \leqslant t\}. \tag{3.2}$$

We now define several graphs, all based on the vertex set N. Thus, we describe them by listing their edge set.

DEFINITION 3.1.

(a) Let l be a positive integer such that $l \leqslant n$. Then

$$G_1(n,l) = \{\{i,j\} : 1 \leqslant i < j \leqslant l\}.$$

(b) Let l be a positive integer such that $2l \leqslant n$. Then

$$G_2(n,l) = \{\{i,j\} : 1 \leqslant i \leqslant l,\ i < j \leqslant n\}.$$

(c) Let l be as in (a). Then

$$B_1(n,l) = \{\{i,j\} : 1 \leqslant i \leqslant j \leqslant l\}.$$

(d) Let l be a positive integer such that $l \leqslant n$. Then

$$B_2(n,l) = \begin{cases} \{\{i,j\} : 1 \leqslant i \leqslant l/2,\ i \leqslant j \leqslant n\} & \text{if } l \text{ is even} \\ \{\{i,j\} : 1 \leqslant i \leqslant [l/2],\ i \leqslant j \leqslant n\} \cup \{1 + [l/2]\} & \text{if } l \text{ is odd.} \end{cases}$$

REMARK 3.1. The following are easily verified:

(a) $|G_1(n,l)| = \binom{l}{2}$, $\nu(G_1(n,l)) = [l/2]$;

(b) $|G_2(n,l)| = \binom{l}{2} + l(n-l)$, $\nu(G_2(n,l)) = l$;

(c) $|B_1(n,l)| = \binom{l+1}{2}$, $\mu(B_1(n,l)) = l$;

(d) Let $m = [l/2]$. Then

$$|B_2(n,l)| = \begin{cases} \binom{m+1}{2} + m(n-m) & \text{if } l \text{ is even}, \\ \binom{m+1}{2} + m(n-m) + 1 & \text{if } l \text{ is odd}, \end{cases}$$

and $\mu(B_2(n,l)) = l$.

LEMMA 3.1 [2]. *Let G be a simple graph on N. Then*

$$n - 2\nu(G) = \max_{X \subset N} \{C_1(G \setminus X) - |X|\}.$$

LEMMA 3.2 [3]. *Let k be a positive integer such that $2k+1 \leqslant n$. Let G be a simple graph on N such that $\nu(G) \leqslant k$. Then*

$$|G| \leqslant \max\{|G_1(n,2k+1)|, |G_2(n,k)|\}. \tag{3.3}$$

If equality holds in (3.3), *then G is isomorphic to $G_1(n,2k+1)$ or $G_2(n,k)$ (or both possibilities can occur if $|G_1(n,2k+1)| = |G_2(n,k)|$).*

We now consider the quantity $u(n,t)$. It turns out that it is crucial in obtaining the maximum dimension of a $\bar{t}$-subspace of $\mathscr{S}_n(F)$ (cf. [6]) and in our characterization of $\bar{t}$-subspaces whose dimension is maximal. Meshulam proved:

THEOREM 3.1 [6]. *Let t be a positive integer such that $t \leqslant n$. Then*

$$u(n,t) = \max\{|B_1(n,t)|, |B_2(n,t)|\}.$$

REMARK 3.2. It follows immediately from Definition 3.1 that

$$u(n,t) = \alpha(n,t).$$

4. MATRIX THEORETIC PRELIMINARIES

We state here two matrix theoretic lemmas that are used in the proof of our main result.

LEMMA 4.1 [4]. *Let t be a positive integer such that $t < n$. Let F be a field such that $|F| > t$, and let L be a $\bar{t}$-subspace of $\mathscr{S}_n(F)$ such that*

$$\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} \in L.$$

Let

$$A = \begin{bmatrix} B & C \\ C^t & D \end{bmatrix} \in L, \qquad \textit{where} \quad B \in \mathscr{S}_t(F).$$

Then

$$C^tC = 0 \quad \textit{and} \quad D = 0.$$

LEMMA 4.2. *Let $r \geqslant 2$, and let F be a field such that $|F| \geqslant r$.*

(a) *Let $a_i \in F$, $i = 1, 2, \ldots, r$, and suppose that the $r \times r$ matrix*

$$A(\lambda) = \begin{bmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & & \\ & & \lambda & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & \\ & & & & & \lambda & 1 \\ a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_{r-1} & a_r \end{bmatrix}$$

is singular for all $\lambda \in F$. Then $a_i = 0$, $i = 1, 2, \ldots, r$.

(b) *Let $a_i \in F$, $i = 1, 2, \dots, r$, and suppose that the $r \times r$ matrix*

$$D_g(\lambda) = \begin{bmatrix}
 & & & & \lambda & 1 & & \\
 & & & & & \lambda & \ddots & \\
 & & & & & & \ddots & 1 \\
 & & & & & & & \lambda \\
1 & & & & & & & \\
\lambda & 1 & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & \lambda & 1 & & & & \\
a_1 & a_2 & \cdots & a_g & a_{g+1} & \cdots & a_{r-1} & a_r
\end{bmatrix}$$

is singular for all $\lambda \in F$. Then $a_i = 0$, $i = 1, 2, \dots, r$.

Proof. (a): This follows immediately from the fact that

$$\det A(\lambda) = \sum_{i=0}^{r-1} (-1)^i a_{i+1} \lambda^i.$$

(b): Appropriate column permutations bring $D_g(\lambda)$ to a matrix of the form $A(\lambda)$. ■

5. $\bar{t}$-SUBSPACES AND GRAPHS

In this section we consider $\bar{t}$-subspaces of $\mathscr{S}_n(F)$. Our purpose is to establish the existence of a basis of special type (to be defined shortly) for every $\bar{t}$-subspace of $\mathscr{S}_n(F)$ of maximal dimension. The tools used are combinatorial, and rely heavily on Meshulam's work. *We assume that $t < n$ and $|F| \geqslant n + 1$.*

DEFINITION 5.1.

(a) Let

$$W_1(n,t) = \{A \in \mathscr{S}_n(F) : a_{ij} = 0 \text{ if } i > t \text{ or } j > t\}.$$

(b) Let

$$W_2(n,t) = \begin{cases} \{A \in \mathscr{S}_n(F) : a_{ij} = 0 \text{ if } i > k \text{ and } j > k\}, & t = 2k, \\ \{A \in \mathscr{S}_n(F) : a_{ij} = 0 \text{ if } i > k \text{ and } j > k, \\ \quad \text{and } (i,j) \neq (k+1, k+1)\}, & t = 2k+1. \end{cases}$$

REMARK 5.1. It is clear that

$$\dim W_1(n,t) = \binom{t+1}{2}$$

and

$$\dim W_2(n,t) = \begin{cases} \binom{k+1}{2} + k(n-k) & \text{if} \quad t = 2k, \\ \binom{k+1}{2} + k(n-k) + 1 & \text{if} \quad t = 2k+1 \end{cases}.$$

Also, $W_i(n,t)$ is a $\bar{t}$-subspace, $i = 1, 2$.

DEFINITION 5.2. Let

$$f_F(n,t) = \max\{\dim W : W \text{ is a } \bar{t}\text{-subspace of } \mathscr{S}_n(F)\}.$$

The following theorem is the main result of [6].

THEOREM 5.1 [6]. $f_F(n,t) = \max\{\dim W_1(n,t), \dim W_2(n,t)\}$.

REMARK 5.2. It is clear from Remark 5.1 that $f_F(n,t) = \alpha(n,t)$.

Meshulam proved Theorem 5.1 by establishing an interesting link between subspaces of $\mathscr{S}_n(F)$ and graphs on N. To describe this link, consider $N \times N$, equipped with the lexicographic ordering. Given any $A \in \mathscr{S}_n(F)$,

$A \neq 0$, let

$$q(A) = \{i_0, j_0\}, \qquad \text{where} \quad (i_0, j_0) = \min\{(i,j) \in N \times N : a_{ij} \neq 0\}. \tag{5.1}$$

Now suppose that $A_i \in \mathscr{S}_n(F)$, $A_i \neq 0$, $i = 1, 2, \dots, m$. We associate with this set of matrices a graph B on the vertex set N whose edges are $q(A_i)$, $i = 1, 2, \dots, m$. A key observation of Meshulam is

$$\max\{\rho(A) : A \in \operatorname{span}\{A_1, A_2, \dots, A_m\}\} \geqslant \mu(B). \tag{5.2}$$

DEFINITION 5.3. Let W be a $\bar{t}$-subspace of $\mathscr{S}_n(F)$.

(a) We say W has a *type I basis* if W has a basis

$$\left\{A_{ij} = \left(a_{pq}^{(ij)}\right) : 1 \leqslant i \leqslant j \leqslant t\right\}$$

which satisfies, for all $1 \leqslant i \leqslant j \leqslant t$,

$$a_{ij}^{(ij)} = a_{ji}^{(ij)} = 1$$

and

$$a_{pq}^{(ij)} = a_{qp}^{(ij)} = 0 \qquad \text{for all } 1 \leqslant p \leqslant q \leqslant t \text{ such that } (p,q) \neq (i,j).$$

(b) We say W has a *type II basis* if the following holds:

(i) If $t = 2k$, then W has a basis

$$\left\{B_{ij} = \left(b_{pq}^{(ij)}\right) : 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant n\right\}$$

which satisfies, for all $1 \leqslant i \leqslant k$, $i \leqslant j \leqslant n$,

$$b_{ij}^{(ij)} = b_{ji}^{(ij)} = 1$$

and

$$b_{pq}^{(ij)} = b_{qp}^{(ij)} = 0 \qquad \text{for all } 1 \leqslant p \leqslant k,\ p \leqslant q \leqslant n \text{ such that } (i,j) \neq (p,q).$$

(ii) If $t = 2k + 1$, then W has a basis

$$\{B_{ij} : 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant n\} \cup \left\{B_{k+1,k+1} = \left(b_{pq}^{(k+1,k+1)}\right)\right\},$$

where B_{ij}, $1 \leqslant i \leqslant k$, $i \leqslant j \leqslant n$ are as in (i), with the additional assumption that their $k+1, k+1$ entry vanishes, and

$$b_{k+1,k+1}^{(k+1,k+1)} = 1,$$

$$b_{pq}^{(k+1,k+1)} = b_{qp}^{(k+1,k+1)} = 0 \qquad \text{for all} \quad 1 \leqslant p \leqslant k, \quad p \leqslant q \leqslant n.$$

We shall show that given an $f_F(n,t)$-dimensional $\bar{t}$-subspace W, there exists a permutation matrix P such that PWP^t has a type I or a type II basis. First, we characterize all graphs B which satisfy $\mu(B) \leqslant t$ and $|B| = u(n,t)$.

THEOREM 5.2. *Let B be a graph on N which satisfies $\mu(B) \leqslant t$ and $|B| = u(n,t)$. Then one of the following holds*:

(a) *B is isomorphic to $B_1(n,t)$.*
(b) *B is isomorphic to $B_2(n,t)$.*
(c) *$t = 2k$, and B is isomorphic to $G_1(n, 2k+1)$.*

Proof. The maximality of $|B|$ implies:

(i) $\mu(B) = t$.
(ii) If $u, v \in N$ and $u \neq v$, and if the loops $\{u\}$ and $\{v\}$ belong to B, so does the edge $\{u, v\}$.

We distinguish two cases.

Case 1. Suppose that $t = 2k$. In this case we have

$$|B_1(n,t)| = \binom{2k+1}{2} \quad \text{and} \quad |B_2(n,t)| = \binom{k+1}{2} + k(n-k).$$

We form from B a *simple* graph G on $N \cup \{n+1\}$ as follows: Any edge of B which is not a loop is also an edge of G. If the loop $\{x\}$ belongs to B, then let $\{x, n+1\} \in G$. So clearly $|G| = |B|$, and $v(G) \leqslant k$. Note also that

$$|G_1(n+1, 2k+1)| = \binom{2k+1}{2} = |B_1(n,t)|$$

and

$$|G_2(n+1,k)| = \binom{k}{2} + k(n+1-k) = \binom{k+1}{2} + k(n-k)$$
$$= |B_2(n,t)|.$$

Suppose that

$$\binom{2k+1}{2} \geqslant \binom{k+1}{2} + k(n-k). \tag{5.3}$$

If strict inequality holds in (5.3), then G is isomorphic to $G_1(n+1, 2k+1)$, by Lemma 3.2. It follows immediately that B is isomorphic to $B_1(n,t)$ or to $G_1(n, 2k+1)$. If equality holds in (5.3), then G is isomorphic to $G_1(n+1, 2k+1)$ or to $G_2(n+1,k)$, by Lemma 3.2. It is clear that (a) or (b) or (c) can occur. Finally, if

$$\binom{2k+1}{2} < \binom{k+1}{2} + k(n-k),$$

similar considerations imply that B is isomorphic to $B_2(n,t)$.

Case 2. Suppose that $t = 2k+1$. The proof of this case is obtained by analyzing the corresponding case in the proof of Theorem 3.1 in [6].

Let G be the simple graph obtained from B by deleting all loops. The assumptions on B imply that $\nu(G) = k$ and G is maximal in the sense that if G' is any graph obtained from G by adding an edge, then $\nu(G') > k$. By Lemma 3.1, there exists $S \subset N$, $|S| = s$, such that $C_1(G \setminus S) = n - 2k + s$.

Denote by $U_1, U_2, \dots, U_{n+s-2k}$ the odd components of $G \setminus S$, and by $Y_1, \dots, Y_l$ the even components of $G \setminus S$. The maximality of G implies that $G[S]$, $G(U_i)$, and $G(Y_j)$ are complete graphs (for all i and j). Also, suppose that $x, y \in N$, $x \neq y$, and $|\{x,y\} \cap S| = 1$. Then $\{x,y\}$ is an edge of G. It follows that $s \leqslant k$. Lemma 3.2 implies that

$$|G| \leqslant \max\left\{\binom{2k+1}{2}, \binom{k}{2} + k(n-k)\right\}.$$

Since

$$|B| = \max\left\{\binom{2k+2}{2}, \binom{k+1}{2} + k(n-k) + 1\right\},$$

B must contain at least $k+1$ loops. The structure of G and the fact that $\mu(B)=2k+1$ imply that there exists a vertex of G which belongs to some U_i and is a loop in B. Without loss of generality we can assume that U_1 contains a vertex which is a loop in B. It follows now that no other U_i can contain a vertex which is a loop in B.

Let $a_i=|U_i|$, $i=1,2,\dots,n+s-2k$, and $b_j=|Y_j|$, $j=1,2,\dots,l$. Using Lemma 2.1, we get

$$\begin{aligned}
|B| &\leqslant \binom{s+1}{2} + s(n-s) + \sum_{j=1}^{l}\binom{b_j+1}{2} + \binom{a_1+1}{2} + \sum_{i=2}^{n+s-2k}\binom{a_i}{2}\\
&\leqslant \binom{s+1}{2} + s(n-s) + \binom{\sum_{j=1}^{l} b_j+1}{2} + \binom{a_1+1}{2} + \sum_{i=2}^{n+s-2k}\binom{a_i}{2}\\
&\leqslant \binom{s+1}{2} + s(n-s) + \binom{\sum_{j=1}^{l} b_j + \sum_{i=1}^{n+s-2k} a_i + 2 + 2k - n - s}{2}\\
&= \binom{s+1}{2} + s(n-s) + \binom{2k-2s+2}{2}.
\end{aligned}$$

Let

$$g(s) = \binom{s+1}{2} + s(n-s) + \binom{2k-2s+2}{2}.$$

This function is strictly convex, since $g''(s)>0$ for all real s. Therefore, its maximum in the interval $[0,k]$ is attained at an end point, so we must have

$$|B| \leqslant \max_{0\leqslant s\leqslant k}\{g(s)\} = \max\{g(0), g(k)\}. \tag{5.4}$$

Hence

$$|B| \leqslant \max\left\{\binom{2k+2}{2}, \binom{k+1}{2} + k(n-k) + 1\right\}. \tag{5.5}$$

But $|B| = u(n, t)$, so we must have equality in (5.5), and consequently, all the inequalities in the chain of inequalities preceding (5.4) must become equalities, and we must also have $s = 0$ or $s = k$.

The equality case in Lemma 2.1 implies that $b_j = 0$, $j = 1, 2, \dots, l$, and $a_i = 1$, $i = 2, \dots, n + s - 2k$. Now, if $s = 0$, then $a_1 = 2k + 1 = t$, so B is isomorphic to $B_1(n, t)$. If $s = k$, then $a_1 = 1$ and B is isomorphic to $B_2(n, t)$. ■

THEOREM 5.3. *Let $t < n$, and let F be a field such that $|F| \geqslant n + 1$ and* char $F \neq 2$. *Let W be an $f_F(n, t)$-dimensional $\bar{t}$-subspace of $\mathscr{S}_n(F)$. Then there exists an $n \times n$ permutation matrix P such that the subspace PWP^t has a type I or type II basis.*

Proof. Let $m = f_F(n, t)$, and let A_i, $i = 1, 2, \dots, m$ be a basis of W. By taking appropriate linear combinations of the basis elements we may assume without loss of generality that $q(A_i) \neq q(A_j)$ for all $i \neq j$.

Consider the graph B on N whose edges are $q(A_i)$, $i = 1, 2, \dots, m$. By (5.2) we must have $\mu(B) \leqslant t$, and we also have

$$|B| = m = f_F(n, t) = \alpha(n, t) = u(n, t).$$

Therefore, B satisfies the assumptions of Theorem 5.2. We claim that (c) of that theorem cannot hold. Indeed, suppose that $t = 2k$ and B is isomorphic to $G_1(n, 2k + 1)$. Then there exists an $n \times n$ permutation matrix P such that $W' = PWP^t$ has a basis $\{A_{ij} = (a_{pq}^{(ij)}) : 1 \leqslant i < j \leqslant 2k + 1\}$ which satisfies, for all $1 \leqslant i < j \leqslant 2k + 1$,

$$a_{ij}^{(ij)} = a_{ji}^{(ij)} = 1,$$

$$a_{pq}^{(ij)} = a_{qp}^{(ij)} = 0 \qquad \text{for all } 1 \leqslant p < q \leqslant 2k + 1 \text{ such that } (p, q) \neq (i, j),$$

and

$$a_{pp}^{(ij)} = 0 \qquad \text{for all} \quad 1 \leqslant p \leqslant i.$$

Let

$$A(\psi_1, \psi_2, \dots, \psi_{k+2}) = \sum_{i=1}^{k} \psi_i A_{2i-1,2i} + \psi_{k+1} A_{2k-1,2k+1} + \psi_{k+2} A_{2k,2k+1}.$$

Let $\alpha = (1, 2, \dots, 2k+1)$ and let $B(\psi_1, \psi_2, \dots, \psi_{k+2}) = A(\psi_1, \psi_2, \dots, \psi_{k+2})[\alpha]$. It can be readily checked that

$$B(\psi_1, \dots, \psi_{k+2}) = B_1 \oplus B_2 \oplus \cdots \oplus B_{k-1} \oplus B_k,$$

where

$$B_1 = \psi_1 \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}, \qquad a \in F,$$

and for $i = 2, \dots, k-1$, B_i is a 2×2 matrix having ψ_i as its off-diagonal entry (while ψ_i does not appear in the main diagonal entries of B_i). B_k is a 3×3 matrix having ψ_k as its 1,2 and 2,1 entries, ψ_{k+1} as its 1,3 and 3,1 entries, and ψ_{k+2} as its 2,3 and 3,2 entries. We choose $\psi_1 \neq 0$ in F so that B_1 is invertible. We may choose $\psi_2 \in F$ so that B_2 is invertible, because the equation $\det B_2 = 0$ is quadratic in ψ_2. Proceeding similarly, we conclude that $\psi_1, \psi_2, \dots, \psi_{k-1}$ can be chosen in F so that $B_1, B_2, \dots, B_{k-1}$ are invertible. For these chosen values of $\psi_1, \psi_2, \dots, \psi_{k-1}$ we have

$$B_k = \begin{bmatrix} a_1 & \psi_k & \psi_{k+1} \\ \psi_k & a_2 + a_3\psi_k + a_4\psi_{k+1} & \psi_{k+2} \\ \psi_{k+1} & \psi_{k+2} & a_5 + a_6\psi_6 + a_7\psi_{k+1} + a_8\psi_{k+2} \end{bmatrix},$$

where $a_i \in F$, $i = 1, \dots, 8$. Then $\det B_k$ is a polynomial of degree 3 in $\psi_k, \psi_{k+1}, \psi_{k+2}$, the coefficient of $\psi_k\psi_{k+1}\psi_{k+2}$ being 2. Since $|F| \geqslant 4$, the values of ψ_k, ψ_{k+1}, and ψ_{k+2} can be chosen in F so that B_k is invertible. This is a contradiction.

It follows that B is isomorphic to $B_1(n, t)$ or to $B_2(n, t)$, and the conclusion follows. ■

6. THE MAIN RESULT

In this section we prove our main result.

THEOREM 6.1. *Suppose that* $0 < t < n$ *and* F *is a field such that* $|F| \geqslant n + 1$ *and* char $F \neq 2$. *Let* W *be an* $f_F(n, t)$*-dimensional* $\bar{t}$*-subspace of* $\mathscr{S}_n(F)$. *Then,* W *is congruent to* $W_1(n, t)$ *or* $W_2(n, t)$.

We shall assume throughout that $|F| \geqslant n + 1$ and char $F \neq 2$. By Theorem 5.3 we may assume that if W satisfies the assumptions of Theorem 6.1, then it possesses a type I or type II basis. Prior to proving Theorem 6.1 we establish several results.

THEOREM 6.2. *Suppose that* $0 < t < n$ *and* F *is a field such that* $|F| \geqslant n + 1$ *and* char $F \neq 2$. *Let* W *be a* $\bar{t}$*-subspace, and suppose that* W *has a type I basis. Then* W *is congruent to* $W_1(n, t)$.

Proof. Let $\{A_{ij} = (a_{pq}^{(ij)}) : 1 \leqslant i \leqslant j \leqslant t\}$ be a type I basis of W. Let $A_0 = \sum_{i=1}^{t} A_{ii}$. Then

$$A_0 = \begin{bmatrix} I_t & C \\ C^t & D \end{bmatrix} \in W.$$

Since W is a $\bar{t}$-subspace, A_0 is congruent to $J = I_t \oplus 0 \in \mathscr{S}_n(F)$. This can be achieved by performing row operations (and the corresponding column operations) from the first t rows (columns) into the last $n - t$ rows (columns). Applying the same congruence to all matrices in W, we may assume that $J \in W$. We shall show that under this assumption $W = W_1(n, t)$. Note that the property of possessing a type I basis is maintained.

Let $1 \leqslant i \leqslant j \leqslant t$, and consider the basis element A_{ij}. Let $t + 1 \leqslant p \leqslant n$. We claim first that

$$a_{pj}^{(ij)} = a_{jp}^{(ij)} = 0 \quad \text{and} \quad a_{pi}^{(ij)} = a_{ip}^{(ij)} = 0. \tag{6.1}$$

So let $B(\lambda) = J + \lambda A_{ij} \in W$ for all $\lambda \in F$. Let $\alpha_p = (1, 2, \ldots, t, p)$, and define $\tilde{m}_p(\lambda) = \det B(\lambda)[\alpha_p]$. Using Lemma 4.1, $\tilde{m}_p(\lambda)$ has a factor λ^2, so we can write $\tilde{m}_p(\lambda) = \lambda^2 m_p(\lambda)$. The assumptions on F imply that $m_p(\lambda) = 0$ for all $\lambda \in F$.

Now, if $i = j$, then $m_p(\lambda)$ is linear in λ. The constant term and the coefficient of λ must vanish, so we get

$$\sum_{\substack{r=1 \\ r \neq i}}^{t} \left(a_{rp}^{(ii)}\right)^2 = 0 \quad \text{and} \quad \sum_{r=1}^{t} \left(a_{rp}^{(ii)}\right)^2 = 0,$$

and therefore $a_{ip}^{(ii)} = 0$.

If $i < j$ then $m_p(\lambda)$ is quadratic in λ. The vanishing of its three coefficients yields the conditions

$$\sum_{\substack{r=1\\ r\neq i,j}}^{t} \left(a_{rp}^{(ij)}\right)^2 = 0,$$

$$a_{ip}^{(ij)} a_{jp}^{(ij)} = 0,$$

and

$$\sum_{r=1}^{t} \left(a_{rp}^{(ij)}\right)^2 = 0.$$

Hence (6.1) holds.

We now show that if $1 \leqslant p \leqslant t$, $p \neq i$, $p \neq j$, and $t+1 \leqslant q \leqslant n$, then $a_{pq}^{(ij)} = a_{qp}^{(ij)} = 0$.

Suppose first that $i = j$. There is nothing to prove if $t = 1$, and the result is immediate if $t = 2$ (using det $A_{ii}[1, 2, q] = 0$). Hence, suppose that $t \geqslant 3$. Let $\{s_1, \ldots, s_{t-2}\} = \{1, 2, \ldots, t\} \setminus \{i, p\}$, and let $\alpha_q = (1, 2, \ldots, t, q)$. Define

$$E(\lambda) = \sum_{l=1}^{t-2} A_{s_l, s_l} + \lambda A_{ii}, \quad \lambda \in F,$$

and let

$$m_q(\lambda) = \det E(\lambda)[\alpha_q].$$

Let $B = \sum_{l=1}^{t-2} A_{s_l, s_l}$. We must have $m_q(\lambda) = 0$ for all $\lambda \in F$, and since $m_q(\lambda) = -\lambda(a_{pq}^{(ii)}\lambda + b_{pq})^2$, we conclude that $a_{pq}^{(ii)} = 0$.

So we may assume that $i < j$. There is nothing to prove if $t = 2$, and the result is immediate if $t = 3$ (using det $A_{ij}[1, 2, 3, q] = 0$). Hence, suppose that $t \geqslant 4$. Let $\{r_1, r_2, \ldots, r_{t-3}\} = \{1, 2, \ldots, t\} \setminus \{i, j, p\}$ and $\alpha_q = (1, 2, \ldots, t, q)$. Define

$$E(\lambda) = \sum_{l=1}^{t-3} A_{r_l, r_l} + \lambda A_{ij}, \qquad \lambda \in F,$$

and let

$$m_q(\lambda) = \det E(\lambda)[\alpha_q].$$

Let $B = \sum_{l=1}^{t-3} A_{r_l, r_l}$. We must have $m_q(\lambda) = 0$ for all $\lambda \in F$, and since $m_q(\lambda) = \lambda^2(a_{pq}^{(ij)}\lambda + b_{pq})^2$, we conclude that $a_{pq}^{(ij)} = 0$.

We have shown that if A is any element of our basis, then $a_{pq} = 0$ if $t + 1 \leqslant p \leqslant n$ or $t + 1 \leqslant q \leqslant n$. The result follows. ■

Next, we deal with $\bar{t}$-subspaces that possess a type II basis. We start with the case that t is even.

THEOREM 6.3. *Suppose that $0 < t = 2k$ and F is a field such that $|F| \geqslant n + 1$ and* char $F \neq 2$. *Suppose that $2.5k + 0.5 \leqslant n$. Let W be a $\bar{t}$-subspace of $\mathcal{S}_n(F)$ which has a basis of type II. Then there exists*

$$Q = \begin{bmatrix} I_k & 0 \\ P & I_{n-k} \end{bmatrix} \in F^{n,n}$$

such that $QWQ^t = W_2(n, 2k)$.

Note that Theorem 6.3 claims that under the assumptions there $W_2(n, 2k)$ is obtained from W by performing elementary row (and corresponding column) operations from the first k rows (columns) into the last $n - k$ rows (columns).

Before proving Theorem 6.3 we state several lemmas.

LEMMA 6.1. *Let $0 < t = 2k < n$, and let W be a $\bar{t}$-subspace of $\mathcal{S}_n(F)$ which has a type II basis. Let*

$$Q = \begin{bmatrix} I_k & 0 \\ P & I_{n-k} \end{bmatrix} \in F^{n,n}$$

and let $W' = QWQ^t$. Then W' has a type II basis.

Proof. Let $\{B_{ij} = (b_{pq}^{(ij)}) : 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant n\}$ be a type II basis of W. Let $B'_{ij} = QB_{ij}Q^t$, $1 \leqslant i \leqslant k$, $i \leqslant j \leqslant n$. Notice that (since t is even) the values of the entries of the $(n - k) \times (n - k)$ principal submatrix in the lower right corner are irrelevant in the definition of basis elements of type II. It is also clear that these entries are the only ones that might change when going from B_{ij} to B'_{ij}, provided that $k + 1 \leqslant j \leqslant n$. Hence, it is clear that appropriate linear combinations of $\{B'_{ij} : 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant n\}$ will produce a type II basis for W'. ■

LEMMA 6.2. *Suppose that the assumptions of Theorem 6.3 are satisfied. Suppose also that $k \geqslant 2$. Let $\{B_{ij} = (b_{pq}^{(ij)}) : 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant n\}$ be a type*

II basis of W. *Let* $1 \leqslant i \leqslant k$, $k+1 \leqslant j \leqslant n$. *Then* $b_{pq}^{(ij)} = b_{qp}^{(ij)} = 0$ *whenever* $p \neq j$, $q \neq j$, $k+1 \leqslant p \leqslant q \leqslant n$.

Proof. Let $p \neq j$, $q \neq j$, $k+1 \leqslant p \leqslant q \leqslant n$. We show $b_{pq}^{(ij)} = 0$. Since $k \geqslant 2$ and $n \geqslant 2.5k + 0.5$, there exist k distinct indices $p_1, p_2, \ldots, p_k$ which satisfy (i) $p_i = j$; (ii) $k+1 \leqslant p_r \leqslant n$, $r = 1, 2, \ldots, k$; (iii) $p_r \notin \{p, q\}$, $r = 1, 2, \ldots, k$. Let

$$C = \sum_{\substack{r=1 \\ r \neq i}}^{k} B_{r, p_r}$$

and let $C(\lambda) = C + \lambda B_{ij}$, $\lambda \in F$. Let

$$\alpha_p = (1, 2, \ldots, k, p, p_1, \ldots, p_k) \quad \text{and} \quad \alpha_q = (1, 2, \ldots, k, q, p_1, \ldots, p_k).$$

$C(\lambda)[\alpha_p | \alpha_q]$ is a $(2k+1) \times (2k+1)$ submatrix of $C(\lambda)$, so its determinant must vanish. But, for every $\lambda \in F$, $\det C(\lambda)[\alpha_p | \alpha_q] = \pm \lambda^2 (\lambda b_{pq}^{(ij)} + c_{pq})$. Hence, $b_{pq}^{(ij)} = 0$. ∎

LEMMA 6.3. *Theorem* 6.3 *holds if* $k = 1$.

Proof. Let $\{B_{1j} = (b_{pq}^{(1j)}) : 1 \leqslant j \leqslant n\}$ be a type II basis of W. Since $\rho(B_{12}) = 2$, it is clear that there exists

$$Q = \begin{bmatrix} 1 & 0 \\ u & I_{n-1} \end{bmatrix} \in F^{n,n}$$

such that $QB_{12}Q^t = E_{12} + E_{21}$. Thus, by Lemma 6.1 we may assume that $J = E_{12} + E_{21} \in W$. We show that under this assumption $W = W_2(n, 2)$.

Let $A \in W$. We show first that $a_{pq} = a_{qp} = 0$ for all $2 < p \leqslant q < n$. Let $\alpha_p = (1, 2, p)$ and $\alpha_q = (1, 2, q)$, and let $C(\lambda) = A + \lambda J$, $m(\lambda) = \det C(\lambda)[\alpha_p | \alpha_q]$, $\lambda \in F$. We must have $m(\lambda) = 0$ for all $\lambda \in F$. It is easy to check that $m(\lambda)$ is quadratic in λ, the coefficient of λ being $-a_{pq}$. Hence $a_{pq} = 0$.

Let $3 \leqslant j \leqslant n$, and consider B_{1j}. Let $C(\lambda) = B_{1j} + \lambda J$. The vanishing of $\det C(\lambda)[1, 2, j]$ for all $\lambda \in F$ implies immediately that $b_{22}^{(1j)} = 0$ and $b_{2j}^{(1j)} = 0$. Let $q \in \{3, \dots, n\} \setminus \{j\}$. The vanishing of $\det C(\lambda)[1, 2, j | 1, 2, q]$ implies that $b_{2q}^{(1j)} = 0$. Hence, $B_{1j} = E_{1j} + E_{j1}$.

Similar considerations show immediately that $B_{11} = E_{11}$. ∎

Proof of Theorem 6.3. The proof is by induction on k. The case $k = 1$ is Lemma 6.3, so we consider the general step. Let $\{B_{ij} = (b_{pq}^{(ij)}) : 1 \leqslant i \leqslant k, i \leqslant j \leqslant n\}$ be a type II basis. We may assume, by Lemma 6.1 and Lemma 6.2 that $B_{1n} = E_{1n} + E_{n1}$.

Let $\hat{W}$ denote that subspace of $\mathscr{S}_{n-2}(F)$ obtained from W by deleting the first and last row, and the first and last column of every $A \in W$. For $A \in W$, let $\hat{A} = A[2, 3, \dots, n-1] \in \hat{W}$. We claim that $\hat{W}$ is a $\overline{2k-2}$-subspace.

So let $D \in \hat{W}$, and let $\rho(D) = r$. Then, there exists a nonsingular $r \times r$ principal submatrix of D, say $D[i_1, i_2, \dots, i_r]$, where $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-2$. Also, $D = \hat{A}$ for some $A \in W$. Let $\alpha = (1, i_1 + 1, i_2 + 1, \dots, i_r + 1, n)$, and let $m(\lambda) = \det(A + \lambda B_{1n})[\alpha]$, $\lambda \in F$. This is quadratic in λ, the coefficient of λ^2 being $-\det D[i_1, i_2, \dots, i_r] \neq 0$. Hence, there exists $\lambda_0 \in F$ such that $\rho(A + \lambda_0 B_{1n}) \geqslant r + 2$. But we also have $\rho(A + \lambda_0 B_{1n}) \leqslant 2k$, so $r \leqslant 2k - 2$, and therefore $\hat{W}$ is a $\overline{2k-2}$-subspace.

We also have

$$\dim \hat{W} \geqslant \dim W - (n + k - 1) = \binom{k+1}{2} + k(n-k) - n - k + 1$$

$$= \binom{k}{2} + (k-1)[(n-2) - (k-1)] = \dim W_2(n-2, 2k-2).$$

Since by assumption $n \geqslant 2.5k + 0.5$, we have $n - 2 \geqslant 2.5k - 1.5 = 2.5(k-1) + 1$. Hence, by Theorem 5.1 and Remark 2.1 we must have $\dim \hat{W} = \dim W_2(n-2, 2k-2)$. Therefore, $\hat{W}$ satisfies the induction hypothesis for $k - 1$, so there exists

$$\hat{Q} = \begin{bmatrix} I_{k-1} & 0 \\ P & I_{n-k-1} \end{bmatrix} \in F^{n-2, n-2}$$

such that $\hat{Q}\hat{W}\hat{Q}^t = W_2(n-2, 2k-2)$. Define now

$$Q = [1] \oplus \hat{Q} \oplus [1] \in F^{n,n}.$$

Q is clearly invertible. We consider the subspace $W' = QWQ^t$ of $\mathscr{S}_n(F)$. By Lemma 6.1, W' has a type II basis. In addition, $a_{pq} = a_{qp} = 0$ for all $k+1 \leqslant p \leqslant q \leqslant n-1$ and every $A \in W'$.

For convenience, we now relabel and write W for W'. So

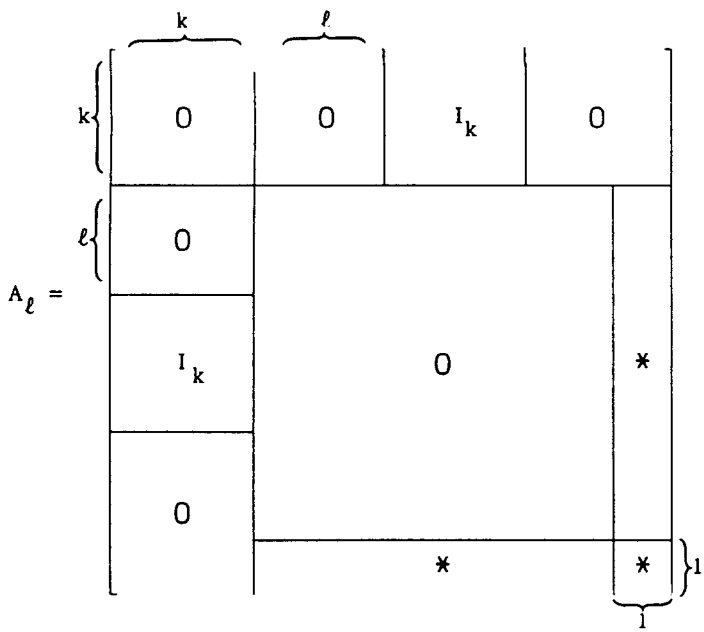
$$a_{pq} = a_{qp} = 0 \qquad \text{for all } k+1 \leqslant p \leqslant q \leqslant n-1 \text{ and every } A \in W. \quad (6.2)$$

Let $\{B_{ij} = (b_{pq}^{(ij)}) : 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant n\}$ be a type II basis for W (recall that we can take $B_{1n} = E_{1n} + E_{n1}$).

Let A_l, $l = 0, 1, 2, \dots, k$, be a sequence of rank-$2k$ matrices in W defined in the following way:

$$A_l = \begin{cases} \displaystyle\sum_{r=1}^{k} B_{r,k+l+r} & \text{if } \ l \leqslant n-2k-1, \\ \displaystyle\sum_{r=1}^{n-l-k-1} B_{r,k+l+r} + \sum_{r=n-l-k}^{k} B_{r,2k+l-n+r+1} & \text{if } \ l \geqslant n-2k, \end{cases}$$

so, if $l \leqslant n-2k-1$,



and if $l \geqslant n - 2k$,

$$A_{\ell} = \left[\begin{array}{cc|ccc|c}
0 & 0 & 0 & 0 & I_{n-\ell-k-1} & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\
0 & 0 & I_{2k+\ell+1-n} & 0 & 0 & 0 \\
\hline
0 & I_{2k+\ell+1-n} & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
I_{n-\ell-k-1} & 0 & 0 & 0 & 0 & * \\
\hline
0\ 0 & \cdots\ 0 & & * & & *
\end{array}\right]$$

In these matrices $*$'s denote entries to be discussed later.

We claim that we can also assume that

$$A_0 = \sum_{r=1}^{k} \left(E_{r,k+r} + E_{k+r,r} \right). \tag{6.3}$$

Indeed, the matrix $\sum_{r=1}^{k}(E_{r,k+r} + E_{k+r,r})$ is obtained from A_0 by elementary row operations (and the corresponding column operations) from rows (columns) $1, 2, \ldots, k$ into the last row (column). These operations amount, of course, to a congruence transformation, and we apply this congruence to W. Note that (6.2) is maintained. For convenience, we relabel and call the resulting subspace W. We now show that $W = W_2(n, 2k)$. It remains to prove that for every $A \in W$,

$$a_{rn} = a_{nr} = 0 \qquad \text{for all} \quad k + 1 \leqslant r \leqslant n. \tag{6.4}$$

Suppose first that $2k + 1 \leqslant r \leqslant n$, and let $A \in W$. Let

$$\alpha_r = (1, 2, \dots, 2k, r), \qquad \gamma = (1, 2, \dots, 2k, n), \qquad A(\lambda) = A + \lambda A_0,$$

$m_r(\lambda) = \det A(\lambda)[\alpha_r | \gamma]$, $\lambda \in F$. Therefore, $m_r(\lambda)$ must vanish for all $\lambda \in F$. The highest power of λ in the expansion of $m_r(\lambda)$ is λ^{2k}, and the corresponding coefficient is $\pm a_{rn}$. Hence (6.4) holds for $2k + 1 \leqslant r \leqslant n$.

Next, we want to show that (6.4) is satisfied by A_l, for all $0 \leqslant l \leqslant k$. The proof is inductive. Clearly, A_0 satisfies (6.4). Let $1 \leqslant l \leqslant k$, and suppose that A_{l-1} satisfies (6.4). We show the same holds for $A_l = (a_{pq}^{(l)})$. Let $A_l(\lambda) = A_l + \lambda A_{l-1}$, $\lambda \in F$. We distinguish two cases:

Case 1. Suppose that $l \leqslant n - 2k - 1$. The structure of A_l and the fact that $\rho(A_l) = 2k$ immediately imply that $a_{k+r,n}^{(l)} = a_{n,k+r}^{(l)} = 0$ for $r = 1, 2, \dots, l$. Let

$$\alpha_l = (1, \dots, k, k + l, \dots, 2k + l - 2, 2k + l, n),$$
$$\gamma_l = (1, \dots, k, k + l, \dots, 2k + l),$$

$m_l(\lambda) = \det A_l(\lambda)[\alpha_l | \gamma_l]$, $\lambda \in F$. This determinant must vanish for all $\lambda \in F$. Letting $a_r = a_{k+l+r,n}^{(l)} = a_{n,k+l+r}^{(l)}$, $r = 0, 1, \dots, k$, it is straightforward to check that

$$m_l(\lambda) = \pm \lambda^{k-1} \det \tilde{A}(\lambda),$$

where $\tilde{A}(\lambda)$ is a $(k + 1) \times (k + 1)$ matrix which looks exactly like the matrix $A(\lambda)$ of Lemma 4.2(a), except that its last row is $(a_0, a_1, \dots, a_k)$. It is clear that $\tilde{A}(\lambda)$ is singular for all $\lambda \in F$. Hence, $a_r = 0$ for all $0 \leqslant r \leqslant k$, by Lemma 4.2. We have shown that A_l satisfies (6.4) in this case.

Case 2. Suppose that $l \geqslant n - 2k$. The structure of A_l and the fact that $\rho(A_l) = 2k$ immediately imply that $a_{3k+l+1-n+r,n}^{(l)} = a_{n,3k+l+1-n+r}^{(l)} = 0$ for $r = 1, 2, \dots, n - 1 - 2k$. Let

$$\alpha_l = (1, \dots, 3k + l - n + 1, l + k, l + k + 1, \dots, n - 2, n),$$

$$\gamma_l = (1, \dots, 3k + l - n + 1, l + k, l + k + 1, \dots, n - 2, n - 1),$$

$$b_r = \begin{cases} a_{k+r,n}^{(l)} = a_{n,k+r}^{(l)}, & r = 1, 2, \dots, l + 2k - n + 1, \\ a_{n-k+r-2,n}^{(l)} = a_{n,n-k+r-2}^{(l)}, & r = l + 2k - n + 2, \dots, k + 1. \end{cases}$$

Let $m_l(\lambda) = \det A_l[\alpha_l|\gamma_l]$, $\lambda \in F$. This determinant must vanish for all $\lambda \in F$, and a straightforward computation shows that

$$m_l(\lambda) = \pm\lambda^{n-1-l-k} \det \tilde{D}(\lambda),$$

where $\tilde{D}(\lambda)$ is a $(k+1) \times (k+1)$ matrix which looks exactly like the matrix $D_g(\lambda)$ of Lemma 4.2(b), with the values $r = k+1$ and $g = l + 2k - n + 1$. Since $\tilde{D}(\lambda)$ must be singular for all $\lambda \in F$, it follows from Lemma 4.2 that $b_r = 0$ for all $1 \leqslant r \leqslant k+1$. Therefore, A_l satisfies (6.4) in this case.

We return to any $A \in W$. Let $l \in \{1, 2, \dots, k\}$ and let $A(\lambda) = A + \lambda A_l$, $\lambda \in F$. Define

$$\alpha_l = \begin{cases} (1, \dots, k, k+l, \dots, 2k+l) & \text{if } l \leqslant n - 2k - 1, \\ (1, \dots, 3k + l - n + 1, k + l, \dots, n-1) & \text{if } l \geqslant n - 2k, \end{cases}$$

$$\gamma_l = \begin{cases} (1, \dots, k, k+l+1, \dots, 2k+l, n) & \text{if } l \leqslant n - 2k - 1, \\ (1, \dots, 3k + l - n + 1, k + l + 1, \dots, n) & \text{if } l \geqslant n - 2k. \end{cases}$$

Then $m_l(\lambda) = \det A(\lambda)[\alpha_l|\gamma_l]$ must vanish for all $\lambda \in F$. The highest power in its expansion is λ^{2k}, and the corresponding coefficient is $a_{k+l,n} = a_{n,k+l}$. Repeating the argument for any $l \in \{1, 2, \dots, k\}$ concludes the proof. ■

The last result required for the proof of the main result is the analogue of Theorem 6.3, when t is odd. The odd case is established by reducing it to the even case. Some of the considerations are similar to preceding ones, so we give less details.

THEOREM 6.4. *Suppose that* $0 < t = 2k + 1$ *and* F *is a field such that* $|F| \geqslant n + 1$ *and* $\operatorname{char} F \neq 2$. *Suppose that* $2.5k + 2.5 \leqslant n$. *Let* W *be a* $\bar{t}$*-subspace of* $\mathcal{S}_n(F)$ *which has a type II basis. Then* W *is congruent to* $W_2(n, t)$.

Proof. We may assume $k > 0$, for the case $t = 1$ is trivial. Let $\{B_{ij} = (b_{pq}^{(ij)}) : 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant n\} \cup \{B_{k+1,k+1} = (b_{pq}^{(k+1,k+1)})\}$ be a type II basis for W. We may apply an appropriate congruence transformation (maintaining a type II basis) so that

$$b_{k+1,r}^{(k+1,k+1)} = b_{r,k+1}^{(k+1,k+1)} = 0 \qquad \text{for all} \quad k+2 \leqslant r \leqslant n. \tag{6.5}$$

So we may assume that (6.5) holds. We claim that $B_{k+1,k+1} = E_{k+1,k+1}$. Let $k+2 \leqslant p \leqslant q \leqslant n$. We show $b_{pq}^{(k+1,k+1)} = 0$. Pick k indices $k+2 \leqslant p_1 < p_2 < \cdots p_k \leqslant n$ so that $\{p, q\} \cap \{p_1, \dots, p_k\} = \varnothing$. Let $C = \sum_{r=1}^{k} B_{r,p_r}$ and let $C(\lambda) = C + \lambda B_{k+1,k+1}$, $\lambda \in F$. Define $\alpha_p = (1, \dots, k, k+1, p, p_1, \dots, p_k)$, $\gamma_q = (1, \dots, k, k+1, q, p_1, \dots, p_k)$. We must have $\det C(\lambda)[\alpha_p|\gamma_q] = 0$ for all $\lambda \in F$. But it is easy to check that

$$\det C(\lambda)\left[\alpha_p|\gamma_q\right] = \pm \det \begin{bmatrix} \lambda + c_{k+1,k+1} & c_{k+1,q} \\ c_{p,k+1} & \lambda b_{pq}^{(k+1,k+1)} + c_{pq} \end{bmatrix}.$$

Hence $b_{pq}^{(k+1,k+1)} = b_{qp}^{(k+1,k+1)} = 0$.

Let $\hat{W}$ be the subspace of $\mathscr{S}_{n-1}(F)$ obtained from W by deleting the $(k+1)$th row and column of every matrix in W. Since $B_{k+1,k+1} = E_{k+1,k+1}$, it is obvious that $\hat{W}$ is a $\overline{2k}$-subspace of $\mathscr{S}_{n-1}(F)$. We also have

$$\dim \hat{W} \geqslant \dim W - (k+1) = \binom{k+1}{2} + k(n-k) + 1 - k - 1$$

$$= \binom{k+1}{2} + k[(n-1) - k] = \dim W_2(n-1, 2k).$$

However, $n \geqslant 2.5k + 2.5$ implies that $n - 1 \geqslant 2.5k + 1.5$, and Theorem 5.1 and Remark 2.1 imply $\dim \hat{W} = \dim W_2(n-1, 2k)$. The conditions of Theorem 6.3 are satisfied, and $\hat{W}$ is congruent (via a congruence as stated in Theorem 6.3) to $W_2(n-1, 2k)$. We may assume from now on that $\hat{W} = W_2(n-1, 2k)$. Thus, for every $A \in W$,

$$a_{pq} = a_{qp} = 0 \qquad \text{for all} \quad k+2 \leqslant p \leqslant q \leqslant n. \tag{6.6}$$

Define now a sequence $C_1, C_2, \dots, C_{k+1}$ of matrices in W as follows:

$$C_l = \begin{cases} \displaystyle\sum_{r=1}^{k} B_{r,k+r+l} & \text{if} \quad l \leqslant n - 2k, \\ \displaystyle\sum_{r=1}^{n-k-l} B_{r,k+l+r} + \sum_{r=n-k-l+1}^{k} B_{r,2k+l-n+r+1} & \text{if} \quad l \geqslant n - 2k + 1. \end{cases}$$

It is clear that $\rho(C_l) = 2k$ for all $1 \leqslant l \leqslant k+1$.

We may perform another congruence transformation on our subspace, namely the following one: Apply elementary row operations (and corresponding column operations) from the first k rows (columns) into the $(k+1)$th row (column). These operations are chosen so that C_1 is transformed to

$$\sum_{r=1}^{k} \left(E_{r,k+r+1} + E_{k+r+1,r} \right).$$

Thus, we may assume throughout that

$$C_1 = \sum_{r=1}^{k} \left(E_{r,k+r+1} + E_{k+r+1,r} \right). \tag{6.7}$$

[Note that (6.6) is maintained after the last congruence.] We have

$$C_1[1,2,\dots,2k+1] = \begin{bmatrix} 0 & 0 & I_k \\ 0 & 0 & 0 \\ I_k & 0 & 0 \end{bmatrix} \in \mathscr{S}_{2k+1}(F),$$

and also $c^{(l)}_{k+1,k+1} = 0$ for all $2 \leqslant l \leqslant k+1$.

We want to show that $W = W_2(n, 2k+1)$. It remains to prove that for all $A \in W$

$$a_{k+1,q} = a_{q,k+1} = 0 \qquad \text{for all} \quad k+2 \leqslant q \leqslant n. \tag{6.8}$$

It follows immediately from (6.7) that $a_{k+1,q} = a_{q,k+1} = 0$ for all $2k+2 \leqslant q \leqslant n$.

Next, we claim that (6.8) is satisfied by C_l for $l = 1, 2, \dots, k+1$. This is certainly the case for $l = 1$, by (6.7). Now let $2 \leqslant l \leqslant k+1$, and suppose that C_{l-1} satisfies (6.8). Consider $C_l + \lambda C_{l-l}$, $\lambda \in F$. Rank considerations and Lemma 4.2 imply that C_l satisfies (6.8).

Finally, let $A \in W$. Let $2 \leqslant l \leqslant k+1$. Consider $A + \lambda C_l$, $\lambda \in F$. It follows that $a_{k+1,k+l} = a_{k+l,k+1} = 0$. ■

Proof of Theorem 6.1. The proof follows immediately from Remark 2.1, Theorem 5.3 and Theorems 6.2, 6.3, and 6.4. More precisely: If $t = 2k$, then W is congruent to $W_1(n,t)$ if $n < 2.5k + 0.5$, and to $W_2(n,t)$ if $n > 2.5k + 0.5$. Both possibilities can occur if $n = 2.5k + 0.5$. If $t = 2k+1$, then W is congruent to $W_1(n,t)$ if $n < 2.5k + 2.5$ and to $W_2(n,t)$ if $n > 2.5k + 2.5$. Both possibilities can occur if $n = 2.5k + 2.5$. ■

REMARK 6.1. It is easy to see that $W_1(n,t)$ cannot be congruent to $W_2(n,t)$, even if their dimensions are equal. This follows from the fact that the intersection of the kernels of the matrices in $W_1(n,t)$ is an $(n-t)$-dimensional subspace, while the intersection of the kernels of the matrices in $W_2(n,t)$ is trivial.

REMARK 6.2. We are indebted to the referee for the following example, which shows that some assumption on $|F|$ is indeed necessary for Theorem 6.1 to hold. Indeed, let $F = Z_2$, and consider the following three $\bar{2}$-subspaces of $\mathscr{S}_3(F)$:

$$U_0 = \{A \in \mathscr{S}_3(F) : a_{11} = a_{22} = a_{33} = 0\};$$

$$U_1 = W_2(3,2);$$

$$U_2 = W_1(3,2).$$

Let $\tilde{R}_1$ denote the set of all matrices in $\mathscr{S}_3(F)$ of rank 1 at most. Then $|\tilde{R}_1 \cap U_i| = 2^i$, $i = 0,1,2$. Therefore, the U_i's are pairwise noncongruent. It is also straightforward to check that the maximal dimension of a $\bar{2}$-subspace of $\mathscr{S}_3(F)$ is 3.

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